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Short coverings in tridimensional spaces arising from sum-free sets

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Abstract

Given a prime power q, define $c(q)$ as the minimum cardinality of a subset H of the tridimensional space $\mathbb{F}_q^3$ which satisfies the following property: every vector in this space differs in at most 1 coordinate from a multiple of a vector in H. On the basis of suitable actions of group, there is established a connection between sum-free sets and corresponding coverings. As an application of our method, there is constructed a class of short coverings which yields $c(q) \leq 3(q+4)/4$, improving the earlier upper bound $c(q) \leq q+1$.

1. Introduction

Considering $\mathbb{F}_q^3$ as a metric space induced by the Hamming distance, note that the ball of center v and radius 1 corresponds to the set $\{v + \beta e_j : \beta \in \mathbb{F}_q, 1 \leq j \leq 3\}$, where e_j denotes a canonical vector. It is natural to ask how many balls are enough to cover the whole space. Kalbfleish and Stanton [2] showed that a minimal covering set uses $\lceil q^2/2 \rceil$ balls.

Nowadays a generalization for higher dimensions is restated in terms of combinatorial coding theory, but these numbers were introduced by Taussky and Todd [6] from a group-theoretical viewpoint almost 60 years ago. As the computation of these numbers has been a hard challenge, several variants and generalizations have been investigated, including the newest ones (see Cohen et al. [1] for an overview).

In particular, a closely related extremal problem is concerned with the following numbers. Throughout this note $\mathbb{F}_q$ denotes the finite field with q elements. Define $c(q)$ as the minimum cardinality of a subset H of $\mathbb{F}_q^3$ which satisfies the following property: every vector in this tridimensional space differs in at most 1 coordinate from a scalar multiple of a vector in H.

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This kind of covering (called a short covering) also allows a generalization for higher dimensions, whose algebraic structure plays a central role and seems to be richer than other kinds of coverings. It is worth mentioning that the equivalence of short coverings comes from actions of a wreath product, according to [3]. It seems interesting to investigate how algebraic concepts are related to short coverings.

We are focused on the above point. By using the action of a direct product, we have established a method for finding short coverings (Theorem 1). A little unexpectedly, this method brings us the main result: a connection between short coverings in $\mathbb{F}_q^3$ and sum-free sets (Theorem 2).

Sum-free sets have been applied to evaluate lower bounds for the classical Ramsey numbers (see [7] for a survey) and have been studied in several contexts and aspects. For instance, recent contributions on the number of sum-free sets in $\{1, 2, \dots, n\}$ have been discussed in [4,5].

It was shown in [3] that $\lceil (q+1)/2 \rceil \le c(q) \le q+1$ for any prime power q. As an application of the main result, it is derived that $c(q) \le 3(q+4)/4$ in Theorem 5. In Section 2 we recall some concepts and notation.

2. Preliminaries

Given the space V_q^3 of all words with length 3 and components from an alphabet of q symbols, the *Hamming distance* $d(u, v)$ between the words u and v is defined as the number of components in which u and v differ. The *sphere* (or *ball*) with center v and radius 1 corresponds to the set

$$B(v) = \{u \in V_q^3 : d(u, v) \le 1\}$$

and let $t = 3q - 2$ denote its cardinality. A subset C is a *covering* of V_q^3 when for every u in V_q^3 there is a word v in C such that $u \in B(v)$. Denote the minimum cardinality of a covering by $K(q)$. The sphere-covering bound gives us the lower bound $K(q) \ge q^3/t \sim q^2/3$. However, the exact value $K(q) = \lceil q^2/2 \rceil$ was established in [2].

A new concept of covering arises when the center h is extended to the scalar multiples of h, as described below. For a vector h in $\mathbb{F}_q^3$, let

$$E(h) = \bigcup_{\alpha \in \mathbb{F}_q} B(\alpha h).$$

A subset H is a *short covering* of $\mathbb{F}_q^3$ when

$$\bigcup_{h \in H} E(h) = \mathbb{F}_q^3. \tag{1}$$

Note that H is a short covering iff $\mathbb{F}_q \cdot H = \{\alpha h : \alpha \in \mathbb{F}_q \text{ and } h \in H\}$ is a "classical" covering of $\mathbb{F}_q^3$. Thus, $c(q)$ is defined as the minimum cardinality of a short covering of $\mathbb{F}_q^3$.

We are going to discuss known bounds on $c(q)$ (see [3]). In contrast to the classical spheres case, the cardinality of $E(h)$ varies according to the vector h. The next statement is similar to the sphere-covering bound:

$$c(q) \ge \frac{q^3 - t}{(q-1)t} \sim q/3. \tag{2}$$

As an immediate application of $c(q) + 1 \le K(q) \le 2c(q) + 1$, the lower bound can be improved to $c(q) \ge \lceil (q+1)/2 \rceil$, by the result of Kalbfleish and Stanton. On the other hand, it is known that $c(q) \le q + 1$. In this work we improve the upper bound to $c(q) \le 3(q+4)/4$.

3. Short coverings and actions

The search for minimal coverings is generally a hard computational problem. One of the steps corresponds to deciding whether a candidate H really satisfies the property in (1), which may involve a lot of operations. In order to decrease the number of operations, it seems interesting to establish a systematic way of finding good coverings. On the basis of invariant sets under suitable actions, our method is described as follows.

Consider the direct product $S_3 \times \mathbb{F}_q^*$, where S_3 denotes the symmetric group of degree 3 and $\mathbb{F}_q^* = \mathbb{F}_q - \{0\}$. For $\varphi \in S_3$, $\lambda \in \mathbb{F}_q^*$, and $v = (v_1, v_2, v_3) \in \mathbb{F}_q^3$, an action of $S_3 \times \mathbb{F}_q^*$ on the set $\mathbb{F}_q^3$ is derived, putting

$$v^{(\varphi,\lambda)} = (\lambda v_{(1)\varphi^{-1}}, \lambda v_{(2)\varphi^{-1}}, \lambda v_{(3)\varphi^{-1}}).$$

In fact, it is clear that if 1 is the identity of $S_3 \times \mathbb{F}_q^*$, then $v^1 = v$ for any $v \in \mathbb{F}_q^3$. Moreover,

$$\begin{aligned} ((v_1, v_2, v_3)^{(\varphi,\lambda)})^{(\sigma,\mu)} &= (\lambda v_{(1)\varphi^{-1}}, \lambda v_{(2)\varphi^{-1}}, \lambda v_{(3)\varphi^{-1}})^{(\sigma,\mu)} \\ &= (\mu\lambda v_{((1)\sigma^{-1})\varphi^{-1}}, \mu\lambda v_{((2)\sigma^{-1})\varphi^{-1}}, \mu\lambda v_{((3)\sigma^{-1})\varphi^{-1}}) \\ &= (v_1, v_2, v_3)^{(\varphi\sigma,\lambda\mu)} \\ &= (v_1, v_2, v_3)^{(\varphi\lambda),(\sigma\mu)} \end{aligned}$$

for every $(v_1, v_2, v_3) \in \mathbb{F}_q^3$ and $(\varphi, \lambda), (\sigma, \mu) \in S_3 \times \mathbb{F}_q^*$.

For $\varphi \in S_3$ and $\lambda \in \mathbb{F}_q^*$ we write $\varphi\lambda$ instead of (φ, λ); thus $v^{\varphi\lambda} = v^{(\varphi,\lambda)}$, $v^{\varphi} = v^{(\varphi,1)}$, and $v^{\lambda} = v^{(1,\lambda)}$. This action also satisfies $v^{\lambda\varphi} = v^{\varphi\lambda}$.

As usual, when a group G acts on a set X, we denote the orbit of v by $v^G = \{v^g : g \in G\}$.

Consider the subset $A = \{(x, y, z) \in \mathbb{F}_q^3 : x, y, z \text{ are pairwise distinct}\}$ of $\mathbb{F}_q^3$ with $q(q-1)(q-2)$ elements and let

$$D = A \cap (\mathbb{F}_q^*)^3 = \{(x, y, z) \in A : x, y, z \neq 0\}.$$

Note that both A and D are invariant under the action of the direct product $S_3 \times \mathbb{F}_q^*$. Thus this action on $\mathbb{F}_q^3$ induces standard actions on both A and D.

Theorem 1. *Let L be a subset of $\mathbb{F}_q^3$ invariant under the action of S_3. Suppose that each orbit of the action of $S_3 \times \mathbb{F}_q^*$ on D contains an element v which can be written as $v = \alpha h + \beta e_j$, for some $h \in L$, $\alpha, \beta \in \mathbb{F}_q$ and $j \in \{1, 2, 3\}$. Thus $L \cup (1, 1, 0)^{S_3}$ is a short covering of $\mathbb{F}_q^3$.*

Proof. Pick an arbitrary vector u in $\mathbb{F}_q^3$. We analyze three cases given by the partition $\{\mathbb{F}_q^3 \setminus A, A \setminus D, D\}$ of $\mathbb{F}_q^3$. In the case where $u \in \mathbb{F}_q^3 \setminus A$, u has at least two coincident coordinates, say $u = (\alpha, \alpha, \beta)$; then $u = \alpha(1, 1, 0) + \beta e_3$ and so $u \in E((1, 1, 0))$. If $u \in A \setminus D$, one of three coordinates must be 0, say $u = (\alpha, \beta, 0)$; then $u = \alpha(1, 1, 0) + (\beta - \alpha)e_2$, that is, u also belongs to $E((1, 1, 0))$. In order to complete the proof, it is enough to show that D is contained in the union of $E(h)$, where h belongs to L. Indeed, given $u \in D$, by hypothesis, there is a v in the orbit of u such that $v = \alpha h + \beta e_j$, for some $h \in L$ and $\alpha, \beta \in \mathbb{F}_q$. Note that $\alpha \neq 0$. Let $\varphi \in S_3$ and $\lambda \in \mathbb{F}_q^*$ be such that $u = v^{\varphi\lambda} = v^{\lambda\varphi}$. Applying λ to v, we get $v^{\lambda} = \alpha' h + \beta' e_j$, for some $\alpha', \beta' \in \mathbb{F}_q$, and now applying φ, we obtain $u = \alpha' h^{\varphi} + \beta' e_j^{\varphi}$. Because L is invariant under S_3, $u \in E(h)$ for some h in L. □

4. Connection between short coverings and sum-free sets

The method above depends on finding suitable sets L. As the main result, there is established below a connection between sum-free sets and short coverings, which allows us to construct a class of short coverings.

We need a few concepts. Given a subset Y in a multiplicative group G, we say that Y is *product-free* in G when for all $a, b \in Y$ we have $ab \notin Y$.

For a family $\mathcal{P} = \{(x_1, y_1), \ldots, (x_k, y_k)\}$ of elements in $G \times G$, put

$$\Delta_{\mathcal{P}} = \{1\} \cup \bigcup_{i=1}^{k} \Delta_{(x_i, y_i)},$$

where $\Delta_{(x,y)}$ denotes $\{x, y, x^{-1}, y^{-1}, xy^{-1}, yx^{-1}\}$.

Theorem 2. *Let $\mathcal{P} = \{(x_1, y_1), \ldots, (x_k, y_k)\}$ be a collection of pairs of elements in $\mathbb{F}_q^*$. If the complement of $\Delta_{\mathcal{P}}$ is product-free in $\mathbb{F}_q^*$, then the set*

$$H = (1, 1, 0)^{S_3} \cup \bigcup_{i=1}^{k} (1, x_i, y_i)^{S_3}$$

is a short covering of $\mathbb{F}_q^3$.

Proof. By Theorem 1, it is sufficient to show that the union $\cup\{E(h) : h \in H\}$ contains a representant of each orbit of the action $G = S_3 \times \mathbb{F}_q^*$ on D. First, we claim that each orbit contains an element of the form $(1, a, b)$ with $a \in \Delta_{\mathcal{P}} \setminus \{1\}$. In fact, for all $(x, y, z) \in D$ we have $(x, y, z)^G = (1, x^{-1}y, x^{-1}z)^G = (1, x^{-1}z, x^{-1}y)^G$ with $x^{-1}z \neq 1$ and $x^{-1}y \neq 1$. Thus, if $x^{-1}y \in \Delta_{\mathcal{P}}$ or $x^{-1}z \in \Delta_{\mathcal{P}}$, we already have the desired result. Otherwise, if both $x^{-1}y$ and $x^{-1}z$ belong to $\mathbb{F}_q^* \setminus \Delta_{\mathcal{P}}$, then $(x^{-1}y)^{-1}, x^{-1}z \in \mathbb{F}_q^* \setminus \Delta_{\mathcal{P}}$ and, since $\mathbb{F}_q^* \setminus \Delta_{\mathcal{P}}$ is product-free, we obtain that $y^{-1}z = (x^{-1}y)^{-1}(x^{-1}z)$ belongs to $\Delta_{\mathcal{P}}$. Now, our assertion follows from the fact that $(1, x^{-1}y, x^{-1}z)^G = (y^{-1}x, 1, y^{-1}z)^G = (1, y^{-1}z, y^{-1}x)^G$.

Let O be an arbitrary orbit and let $(1, a, b)$ be an element of O with $a \in \Delta_{\mathcal{P}} \setminus \{1\}$. Thus $a \in \{x_i, y_i, x_i^{-1}, y_i^{-1}, x_i^{-1}y_i, y_i x_i^{-1}\}$ for some i, $1 \leq i \leq k$. We have to analyze six cases:
if $a = x_i$, then $(1, a, b) = (1, x_i, y_i) + (b - y_i)(0, 0, 1)$;
if $a = x_i^{-1}$, then $(1, a, b) = x_i^{-1}(x_i, 1, y_i) + (b - x_i^{-1}y_i)(0, 0, 1)$;
if $a = x_i^{-1}y_i$, then $(1, a, b) = x_i^{-1}(x_i, y_i, 1) + (b - x_i^{-1})(0, 0, 1)$.

The other three cases are similar to those above. Therefore $(1, a, b) \in \cup\{E(h) : h \in H\}$ in any case and the proof follows by Theorem 1. □

Let $(G, +)$ be an abelian group. A subset S of G is *sum-free* if every sum of two arbitrary elements of S does not belong to S, that is, $(S + S) \bigcap S = \emptyset$, where $S + S = \{a + b : a \in S \text{ and } b \in S\}$.

The concepts of the product-free set and the sum-free set are closely related, according to the remark below.

Remark 3. Let ξ be a generator of the cyclic group $\mathbb{F}_q^*$ and take the isomorphism $\mathbb{F}_q^* \to \mathbb{Z}_{q-1}$ given by $\xi^i \mapsto i$. By this isomorphism, a product-free set in $\mathbb{F}_q^*$ can be translated to a sum-free set in $\mathbb{Z}_{q-1}$, and conversely. Considering the additive group instead of the multiplicative one, the set $\Delta_{(\xi_i^x, \xi_i^y)}$ corresponds to the set $\Delta_{(x,y)} = \{x, y, -x, -y, x - y, y - x\}$. In addition, given a family $\mathcal{P} = \{(x_1, y_1), \ldots, (x_k, y_k)\}$ of pairs of elements in $\mathbb{Z}_{q-1}$, $\Delta_{\mathcal{P}}$ becomes the set formed by the union of $\{0\}$ and the k subsets $\Delta_{(x_i, y_i)}$ of $\mathbb{Z}_{q-1}$, where $1 \le i \le k$. The additive notation was chosen in the following results.

Example 4. We illustrate the construction for $q = 13$. Choose the singleton family $\mathcal{P} = \{(10, 6)\} \subset \mathbb{Z}_{12} \times \mathbb{Z}_{12}$. Evaluating $\Delta_{(10,6)}$, we obtain $\Delta_{\mathcal{P}} = \{0, 2, 4, 6, 8, 10\}$, whose complement in $\mathbb{Z}_{12}$ is sum-free. By Remark 3 and Theorem 2, the set $\{(1, 1, 0), (0, 1, 1), (1, 0, 1)\} \cup (1, \xi^{10}, \xi^{6})^{S_3}$ is a short covering of $\mathbb{F}_q^3$ for any generator ξ of $\mathbb{F}_q^*$, and so $c(13) \le 9$.

5. Upper bound for $c(q)$

For the proof of the next result we use the following notation. As usual, the cardinality of a set X is denoted by $|X|$. Let n be an integer with $n \ge 4$. Given integers k, r and s such that $0 < r$, $s < k < n$, where r is even and s is odd, write

$$\mathcal{A}(k, s) = \{(1, k), (3, k-2), \ldots, (s, k-(s-1))\},$$
$$\mathcal{B}(k, s) = \{(1, k-1), (3, k-3), \ldots, (s, k-s)\},$$
$$\mathcal{C}(r) = \{(4.1-2, 4.2-2), (4.3-2, 4.4-2), \ldots, (4(r-1)-2, 4.r-2)\},$$
$$\mathcal{D}(r) = \{(2, 4), (6, 8), \ldots, (r-2, r)\},$$

where $\mathcal{A}(k, s)$, $\mathcal{B}(k, s)$, $\mathcal{C}(r)$ and $\mathcal{D}(r)$ are subsets of $\mathbb{Z}_n \times \mathbb{Z}_n$. We also write $\mathcal{C}(0) = \mathcal{D}(0) = \emptyset$. Moreover, a subset S of $\mathbb{Z}_n$ is *symmetric* if $-s \in S$ whenever $s \in S$, that is, $S = -S$, where $-S = \{-s | s \in S\}$. The complement of S in $\mathbb{Z}_n$ is denoted by $\bar{S}$.

Theorem 5. *Given a prime power $q = p^r$, let $n = q - 1$. For every $q \ge 5$ we obtain*

(1) *$c(q) \le 3(q+3)/4$ if q is odd and 4 divides n;*
(2) *$c(q) \le 3(q+1)/4$ if q is odd and 4 does not divide n;*
(3) *$c(q) \le 3(q+4)/4$ if $p = 2$.*

Proof. The proof is an application of Theorem 2. We have to present suitable collections $\mathcal{P}$. Let ξ denote a generator of $\mathbb{F}_q^*$ and consider the additive group $\mathbb{Z}_n$. For part (1), if $n/4$ is even, put

$$\mathcal{P} = \mathcal{D}\left(\frac{n}{2}\right);$$

otherwise, put

$$\mathcal{P} = \mathcal{D}\left(\frac{n}{2} - 2\right) \cup \left\{\left(\frac{n}{2}, \frac{n}{2}\right)\right\}.$$

In each case, we have $\Delta_{\mathcal{P}} \subset \{2, 4, \ldots, n\}$ and since $\Delta_{\mathcal{P}} = -\Delta_{\mathcal{P}}$ it is not difficult to check that $\Delta_{\mathcal{P}} = \{2, 4, \ldots, n\}$. Therefore $\overline{\Delta_{\mathcal{P}}} = \{1, 3, \ldots, n-1\}$ which is sum-free in $\mathbb{Z}_n$. In order to evaluate $c(q)$, we initially observe that if $n/4$ is even, then $\mathcal{P}$ contains $n/8$ elements. Thus, it follows from Remark 3 and Theorem 2 that

$$c(q) \le |(1, 1, 0)^{S_3}| + |(1, \xi^2, \xi^4)^{S_3}| + \cdots + |(1, \xi^{\frac{n}{2}-2}, \xi^{\frac{n}{2}})^{S_3}| = 3 + 6\frac{n}{8} = \frac{3q+9}{4}.$$

Now if $n/4$ is odd, using again Theorem 2, we obtain

$$c(q) \le |(1, 1, 0)^{S_3}| + |(1, \xi^2, \xi^4)^{S_3}| + \cdots + |(1, \xi^{\frac{n}{2}-4}, \xi^{\frac{n}{2}-2})^{S_3}| + |(1, \xi^{\frac{n}{2}}, \xi^{\frac{n}{2}})^{S_3}|,$$

that is,

$$c(q) \leq 3 + 6(n-4)/8 + 3 = (3q+9)/4,$$

proving (1).

For part (2), if $4|(\frac{n}{2}-1)$ put

$$\mathcal{P} = \mathcal{D}\left(\frac{n}{2}-1\right);$$

otherwise, put

$$\mathcal{P} = \mathcal{D}\left(\frac{n}{2}-3\right) \cup \left\{\left(\frac{n}{2}-1, \frac{n}{2}-1\right)\right\}.$$

Arguing as above, the upper bound for $c(q)$ also follows when q is odd and 4 does not divide n.

For part (3), we remark that $n = 2^r - 1$ is congruent to 0 or 1 modulo 3. Suppose first that $n = 3k$, for some integer k. Observe that k is odd and since $n = 2^r - 1$, with $r \geq 4$, there exists an integer l such that $k = 4l + 1$; moreover l is odd. Since the set $S = \{k+1, k+2, \ldots, 2k-1\}$ is sum-free in $\mathbb{Z}_n$, by Theorem 2, it is enough to show that $\overline{\Delta_{\mathcal{P}}} \subset S$ for a suitable family $\mathcal{P}$. Put

$$\mathcal{P} = \mathcal{A}\left(k, \frac{k-3}{2}\right) \cup \left\{\left(\frac{k+1}{2}, \frac{k+1}{2}\right)\right\} \cup \mathcal{C}(l-1) \cup \{(4l-2, 4l-2)\}.$$

Notice that $\bar{S} = \{0\} \cup X \cup -X$, where $X = \{1, 2, \ldots, k\}$. Furthermore, $X = U \cup V \cup W$, where

$$U = \{1, 3, 5, \ldots, k\}, \quad V = \{4.1, 4.2, \ldots, 4.l\}, \quad W = \{4.1-2, 4.2-2, \ldots, 4.l-2\}.$$

By construction, $a, b, b-a \in \Delta_{\mathcal{P}}$ whenever $(a, b) \in \mathcal{P}$. Thus, as $\mathcal{A}(k, (k-3)/2) \cup \{((k+1)/2, (k+1)/2)\} \subset \mathcal{P}$, we get

$$U \subset \Delta_{\mathcal{P}} \quad \text{and} \quad V = \{b-a | (a, b) \in \mathcal{A}(k, (k-3)/2)\} \subset \Delta_{\mathcal{P}}.$$

The inclusion $W \subset \Delta_{\mathcal{P}}$ follows from $\mathcal{C}(l-1) \cup \{(4l-2, 4l-2)\} \subset \mathcal{P}$. Hence $X \subset \Delta_{\mathcal{P}}$, and since $\Delta_{\mathcal{P}}$ is symmetric, we obtain $\bar{S} \subset \Delta_{\mathcal{P}}$. Thus $\overline{\Delta_{\mathcal{P}}}$ is a sum-free set in $\mathbb{Z}_n$. Therefore, using Remark 3 and Theorem 2, $\mathcal{P}$ induces a short covering of $\mathbb{F}_q^3$. It remains to estimate the cardinality of this covering. We have

$$c(q) \leq 3 + 6\left(\frac{k-1}{4}\right) + 3 + 6\left(\frac{l-1}{2}\right) + 3 = \frac{3}{4}(q+4).$$

Finally, let us assume that n is congruent to 1 modulo 3, that is, $n = 3k+1$ for some integer k. In this case, $S = \{k+1, k+2, \ldots, 2k\}$ is sum-free and symmetric in $\mathbb{Z}_n$. Note that $\bar{S} = \{0\} \cup X \cup -X$, where $X = \{1, 2, \ldots, k\}$. Since k is even and $n = 2^r - 1$, $k = 4l + 2$ for some even integer l. Let

$$\mathcal{P} = \mathcal{B}\left(k, \frac{k}{2}\right) \cup \mathcal{C}(l) \cup \{(4l+2, 4l+2)\}.$$

It is not difficult to see that $\bar{S} \subset \Delta_{\mathcal{P}}$, and so $\overline{\Delta_{\mathcal{P}}}$ is sum-free in $\mathbb{Z}_n$. Thus $\mathcal{P}$ induces a short covering of $\mathbb{F}_q^3$. A routine calculation shows that $c(q) \leq 3(q+4)/4$. This concludes the proof. □

6. Final remarks

We conclude this paper with a few remarks about the construction employed.

The problem of evaluating $c(q)$ seems to be more difficult than that of evaluating the classical $k(q)$. A new obstacle arises here: besides the size of $E(h)$ being not invariant, these sets are highly intersecting.

However, the above method constitutes a systematic way of constructing short coverings. In order to avoid a large number of tests, we attempted to explore symmetries of suitable short coverings, by using their algebraic properties.

The table below shows bounds on $c(q)$ for small entries. The exact values come from [3]. For the left cases, the corresponding lower bounds follow from $c(q) \geq \lceil (q+1)/2 \rceil$, which give us better estimates than (2). The upper bounds are derived from Theorem 5, except for the case $q = 13$.

Indeed, Theorem 5 implies $c(13) \leq 12$, while the sharper bound $c(13) \leq 9$ can be derived from Example 4. This illustrates that our upper bound is not tight for this case.

q	2	3	4	5	7	8	9	11	13
$c(q)$	1	3	3	3–6	4–6	5–9	5–9	6–9	7–12

Any improvement on lower or upper bound would be very desirable. The asymptotic behaviour of $c(q)$ is still an open problem.

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